

Crossing Number for Graphs with Bounded Pathwidth*

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Abstract

The crossing number is the smallest number of pairwise edge-crossings when drawing a graph into the plane. There are only very few graph classes for which the exact crossing number is known or for which there at least exist constant approximation ratios. Furthermore, up to now, general crossing number computations have never been successfully tackled using bounded width of graph decompositions, like treewidth or pathwidth.

In this paper, we for the first time show that crossing number is tractable (even in linear time) for maximal graphs of bounded pathwidth 3. The technique also shows that the crossing number and the rectilinear (a.k.a. straight-line) crossing number are identical for this graph class, and that we require only an $O(n) \times O(n)$ -grid to achieve such a drawing.

Our techniques can further be extended to devise a 2-approximation for general graphs with pathwidth 3, and a $4w^3$ -approximation for maximal graphs of pathwidth w . This is a constant approximation for bounded pathwidth graphs.

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1 Introduction

The crossing number $cr(G)$ is the smallest number of pairwise edge-crossings over all possible drawings of a graph G into the plane. Despite decades of lively research, see e.g. [22, 23], even most seemingly simple questions—like the crossing number of complete or complete bipartite graphs—are still open, cf. [20]. There are only very few graph classes, e.g., Peterson graphs $P(3, n)$ or Cartesian products of small graphs with paths or trees, see [3, 18, 21], for which the crossing number is known or can be efficiently computed. Considering approximations, we know that computing $cr(G)$ is APX-hard [4], i.e., there does not exist a PTAS. However, the best known approximation ratio for general graphs with bounded maximum degree is $\tilde{O}(n^{0.9})$ [9]. We only know constant approximation ratios for special graph classes. In fact, all known constant approximation ratios are based on one of two concepts: *Topology-based*

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approximations require that G can be embedded without crossings on a surface of some fixed or bounded genus [12, 14, 15]. *Insertion-based* approximations assume that there is only a small (i.e., bounded size) subset of graph elements whose removal leaves a planar graph [5–8]. In either case, the ratios are constant only if we further assume bounded maximum degree.

While treewidth and pathwidth have been very successful tools in many graph algorithm scenarios, they have only very rarely been applied to crossing number: Since general crossing number seems not to be describable with second order monadic logic, Courcelle’s result [10] regarding treewidth-based tractability can only be applied if cr itself is bounded [13, 16].

Contribution. In this paper, we for the first time show that such graph decompositions, in our case pathwidth, *can* be used for crossing number. We show for maximal graphs G of pathwidth 3 (see Section 3):

- We can compute the *exact* crossing number $cr(G)$ in linear time.
- The topological $cr(G)$ equals the *rectilinear* crossing number $\overline{cr}(G)$, i.e., the crossing number under the restriction that all edges need to be drawn as straight lines.
- We can compute a drawing realizing $\overline{cr}(G)$ on an $O(n) \times O(n)$ -grid.

We then generalize these techniques to show:

- A 2-approximation for $cr(G)$ and $\overline{cr}(G)$ for general graphs of pathwidth 3, see Section 4.
- A $4w^3$ -approximation for $cr(G)$ for maximal graphs of pathwidth w , see Section 5. This can be achieved by placing vertices and bend points on a $4n \times wn$ grid.

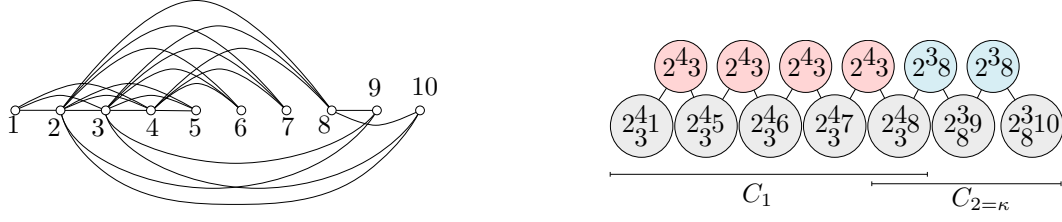
Observe that these are the first crossing number approximation ratios known that are *not* dependent on the graph’s maximum degree. As a complementary side note, we show that the *weighted* (possibly rectilinear) crossing number is weakly NP-hard already for maximal graphs with pathwidth ≥ 4 in Section 2.

2 Preliminaries

We always consider a simple undirected graph G with n vertices as our input. A drawing of G is a mapping φ of vertices and edges to points and simple curves in the plane, respectively. The curve $\varphi(e)$ of an edge $e = (u, v)$ does not pass through any point $\varphi(w)$, $w \in V(G)$, but has its ends at $\varphi(u)$ and $\varphi(v)$. When asking for a crossing minimum drawing of G , we can restrict ourselves to *good* drawings, which means that adjacent edges do not cross, non-adjacent edges cross at most once, and no three edges cross at the same point of the drawing. For other drawings, straight-forward redrawing arguments, see e.g. [22], show that the crossing number can never increase when establishing these properties.

A *clique* is a complete graph and a *biclique* is a complete bipartite graph. While the exact crossing number is unknown for general cliques and bicliques, there are upper bound constructions, conjectured to attain the optimal value. In particular the old construction due to Zarankiewicz, attaining $\lfloor \frac{n_1}{2} \rfloor \lfloor \frac{n_1-1}{2} \rfloor \lfloor \frac{n_2}{2} \rfloor \lfloor \frac{n_2-1}{2} \rfloor$ crossings for K_{n_1, n_2} , is known to give the optimum for $n_1 \leq 6$ [17].

A prominent variant of the traditional (“topological”) crossing number $cr(G)$ is the *rectilinear* crossing number $\overline{cr}(G) \geq cr(G)$, sometimes also known as geometric or straight-line crossing number. Thereby, edges are required to be drawn as straight line segments without any bends. Interestingly, while we know $\overline{cr}(G) > cr(G)$ in general (e.g., already for complete graphs), Zarankiewicz’s construction is a straight-line drawing, suggesting that maybe $cr(G) = \overline{cr}(G)$ for bicliques.



■ **Figure 1** (left) A graph, with vertices in age order according to \mathcal{P} . (right) Its alternating path decomposition \mathcal{P} of width 3, with two clusters: C_1 has $T(C_1) = \{2, 3, 4\}$, and consists of all bags containing this anchor-triplet. Analogously, we have $T(C_2) = \{2, 3, 8\}$. In C_1 , the lost vertex is $x_1^- = 1$ and the emerging vertex is $x_1^+ = 8$.

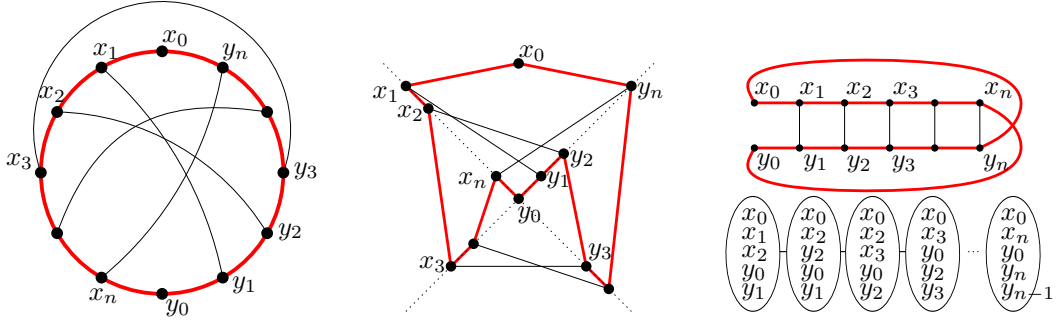
Alternating path decompositions and clusters. There are several equivalent definitions of pathwidth; we use here the one based on tree decompositions, see e.g. [19]. A *path decomposition* \mathcal{P} of a connected graph G consists of a finite set of *bags* $\{X_i \mid 1 \leq i \leq \xi \in \mathbb{N}\}$, where each bag is a subset of the vertices of G , such that for every edge (v, w) at least one bag contains both v and w , and for every vertex v of G the set of bags containing v forms an interval (i.e., the underlying graph formed by the bags is a path). The indexing of the bags gives a total ordering and we may speak of *first*, *last*, *preceding*, and *succeeding* bags. The *width* of a path decomposition is the maximum cardinality of a bag minus one, i.e., $\max_{1 \leq i \leq \xi} |X_i| - 1$. The *pathwidth* $\mathbf{w} := \mathbf{w}(G)$ of G is the smallest width that can be achieved by a path decomposition of G . A *maximal pathwidth- \mathbf{w} graph* is a graph of pathwidth \mathbf{w} for which adding any edge increases its pathwidth. In particular, this implies that the vertices in each bag form a clique. We assume that $n > \mathbf{w} + 1$; otherwise G is a clique and the crossing number is 0 for $\mathbf{w} = 3$ and easily approximated within a factor of $O(1)$ for bigger \mathbf{w} .

Several additional constraints can be imposed on the bags and the path decomposition without affecting the required width. We use a reduced variant of a *nice* path decomposition that we call an *alternating* path decomposition (see Fig. 1 and [19]):

- There are exactly $\xi = 2n - 2\mathbf{w} - 1$ bags.
- $|X_i| = \mathbf{w} + 1$ if i is odd and $|X_i| = \mathbf{w}$ if i is even.
- For any even $1 < i < \xi$, we have $X_{i-1} \supset X_i \subset X_{i+1}$.

Note that for any odd i there is exactly one vertex v that is in X_i but not in bag X_{i+1} . We say that v is *forgotten* by bag X_{i+1} . Similarly, bag X_i contains exactly one vertex v that was not in bag X_{i-1} . We say that v is *introduced* by bag X_i . We define the *age-order* $\{v_1, \dots, v_n\}$ of the vertices of G as follows: v_1 is forgotten by X_2 ; $v_2, \dots, v_{\mathbf{w}+1}$ are the other vertices of bag X_1 in arbitrary order. The order of the remaining vertices corresponds to the order of the bags by which they are introduced. We say that v_i is *older* than v_j if $i < j$, so the three oldest vertices are v_1, v_2, v_3 . Note that we can choose v_2, v_3 arbitrarily among $X_1 - \{v_1\}$. In particular, if two vertices $p, q \in X_1$ are specified, then we can ensure that they are among the three oldest; this will be exploited in Section 4.2.

In our algorithms and proofs, we will work with special subsets of bags called *clusters*. Let G be a connected graph of pathwidth 3 with an alternating path decomposition $\mathcal{P} = \{X_i\}_{1 \leq i \leq \xi}$. Consider a set of three vertices Y that constitute at least one bag (this bag has an even index). There can be several such bags with exactly those vertices, but all bags containing Y are consecutive. For any such Y , we define a *cluster* C as the maximal consecutive set of bags that all contain Y . We say that $T(C) := Y$ is the *anchor-triplet* of C . Any cluster has at least 3 bags. They alternate between size 4 and 3, starting and ending with size-4 bags. Two consecutive clusters overlap in exactly one bag (which consequently



■ **Figure 2** (left) A drawing of G for $n = 4$. Edges of Q are bold red. (center) An equivalent straight-line drawing. (right) G , viewed as Möbius-strip, with a path decomposition of width 4.

has size 4). The order of the bags induces a unique order of the clusters $\{C_1, \dots, C_\kappa\} =: \mathcal{C}$.

Note that a cluster C can be described as a set of bags, or by its anchor-triplet. Denote the vertices that appear in the union of bags of C by $V(C)$, and let $n(C) := |V(C)|$. The following observation is trivial (because any vertex of the anchor-triplet of C belongs to all bags of C) but crucial for our analysis.

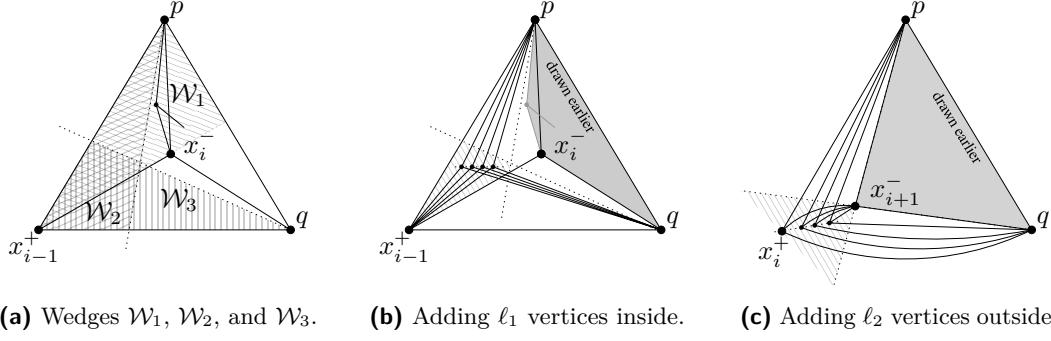
► **Observation 1.** *Let G be a maximal pathwidth-3 graph and let C be a cluster. Then the graph induced by $V(C)$ consists of the triangle induced by $T(C)$ and (edge-disjoint) a biclique $K_{3, n(C)-3}$ with one partition being $T(C)$.*

We define the *emerging vertex* of C_i , denoted by x_i^+ , as the vertex introduced by the last bag of C_i . Note that x_i^+ belongs to the anchor-triplet of the next cluster C_{i+1} if $i < \kappa$. We define the *lost vertex* of C_i , denoted by x_i^- , as the vertex that was forgotten by the second bag of C_i . Note that x_i^- belongs to the anchor-triplet of the previous cluster C_{i-1} if $i > 1$, but not to the anchor-triplet of C_i . Observe that $x_1^- = v_1$, $x_\kappa^+ = v_n$, $x_{i-1}^+ \neq x_i^-$ and $T(C_i) = T(C_{i-1}) \cup \{x_{i-1}^+\} \setminus \{x_i^-\}$ for all $2 \leq i \leq \kappa$. For notational simplicity, we define $x_0^+ := v_2$. Any vertex x that belongs to C_i but is not in $T(C_i) \cup \{x_i^+, x_i^-\}$ is called a *singleton* of C_i . Vertex x belongs to a “middle” bag of C_i and only appears in this bag; it belongs to no cluster other than C_i . See Fig. 1 for an example.

NP-hardness. To complement the results of our paper, we can show that:

► **Theorem 2.** *The weighted and weighted rectilinear crossing number problems are weakly NP-hard already for (maximal) pathwidth-4 graphs that have non-weighted crossing number 1.*

Proof. (Sketch, see appendix for details.) For the weighted crossing number, a crossing contributes a crossing weight of $w_1 \cdot w_2$ if two edges with weights w_1, w_2 cross. We can follow the ideas of [5], see also Fig. 2. Given an instance $\mathcal{I} := \{a_i \in \mathbb{N}\}_{1 \leq i \leq n}$ of PARTITION, we define a graph G on $2n + 2$ vertices $x_0, \dots, x_n, y_0, \dots, y_n$ that form a $(2n+2)$ -cycle Q in this order. Edges of Q have weight S^2 for $S := \frac{1}{2} \sum_i a_i$. Additionally, there are n edges $e_i = (x_i, y_i)$, $1 \leq i \leq n$; edge e_i has weight a_i . One can show that instance \mathcal{I} has a solution if and only if G has a (straight-line) drawing with total crossing weight at most $S^2 - \frac{1}{2} \sum_i a_i^2$. We observe that G is a Möbius wheel which is known to have pathwidth 4. We can make G maximal by adding edges of weight 0 or small ε . ◀



■ **Figure 3** Drawing maximal pathwidth-3 graphs. (For ease of legibility we draw some edges in (c) slightly curved.)

3 Exact Algorithm for Maximal Pathwidth-3 Graphs

Let G be a maximal pathwidth-3 graph and fix an alternating path decomposition of width 3. By maximality, all bags form cliques, and in particular, each anchor-triplet induces a triangle in the graph, called *anchor triangle* consisting of *anchor edges*.

The general idea to draw G is to iterate through the clusters C_1, \dots, C_κ . When considering cluster C_i , its first bag will already be drawn and the anchor triangle will form the outer face of the current drawing. About half of the vertices introduced by C_i will be drawn inside the anchor triangle while the other half will be drawn outside. The number of crossings that these vertices add will be exactly the minimum number of crossings needed to draw the biclique $K_{3, n(C_i)-3}$ of cluster C_i , hence leading to an optimal drawing.

We start with drawing bag $X_1 = \{v_1, v_2, v_3, v_4\}$ as a planar drawing of K_4 with the vertices $T(C_1) = X_2 = \{v_2, v_3, v_4\}$ on the outer face. Now we iterate over all clusters C_i , $1 \leq i \leq \kappa$, drawing their bags with the following invariants:

- The drawing is good and straight-line.
- Before drawing C_i , the outer face contains the three vertices $T(C_i)$.
- For any $j \leq i$, the anchor edges of C_j are drawn without crossings.

Let ℓ be the number of singleton vertices in C_i (possibly $\ell = 0$). We need to place the singletons and the emerging vertex. We will add $\ell_1 := \lfloor (\ell + 1)/2 \rfloor \leq \ell$ vertices into an inner face of the current drawing and $\ell_2 = \lceil (\ell + 1)/2 \rceil \geq 1$ vertices on the outside as follows:

Placement on the inside. By the invariant the outer face consists of the edges connecting $T(C_i) = \{x_{i-1}^+, p, q\}$ for some p, q . W.l.o.g. assume that x_{i-1}^+ , p , and q occur in clockwise order walking along the outer face. By maximality, and because x_{i-1}^+ has just been introduced, x_{i-1}^+ has degree 3 in the current graph, and its neighbors are p, q, x_i^- .

Let \mathcal{R} be the open region obtained by the intersection of three open “wedges” $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ defined as follows: Wedge \mathcal{W}_1 emanates from x_{i-1}^+ between edges (x_{i-1}^+, p) and (x_{i-1}^+, x_i^-) in the interior of the triangle induced by $T(C_i)$. Wedge \mathcal{W}_2 (\mathcal{W}_3) emanates at p (q) inside of $T(C_i)$ and runs along edge (p, x_{i-1}^+) ((q, x_{i-1}^+) , respectively) with a sufficiently small angle such that it crosses only edges incident to x_{i-1}^+ . Any point inside \mathcal{R} can be connected to all of p, q, x_{i-1}^+ with straight lines and a single crossing (with edge (x_{i-1}^+, x_i^-)).

Consider a straight line s through \mathcal{R} but not through any of p, q, x_{i-1}^+ . Place ℓ_1 vertices (for ℓ_1 singletons of C_i) along s within \mathcal{R} , and connect each of them to all of p, q, x_{i-1}^+ . All generated crossings are with edge (x_{i-1}^+, x_i^-) or among the added edges. The drawing is straight-line and good (no three edges cross in a point), and the number of added crossings is $\ell_1 + \binom{\ell_1}{2} = \frac{1}{2}\ell_1(\ell_1 + 1)$.

Placement on the outside. The outer face of the drawing is still formed by the edges connecting $T(C_i)$, since all vertices from the paragraph above were added inside \mathcal{R} and thus in the interior of $T(C_i)$. We know that the vertex x_{i+1}^- in $T(C_i)$ will be lost in the next cluster C_{i+1} (if there is any); it will play a prominent role now. Since we may or may not have $x_{i+1}^- = x_{i-1}^+$, we label the vertices of $T(C_i)$ afresh as $\{x_{i+1}^-, p', q'\}$.

Define an open wedge \mathcal{W} in the exterior of $T(C_i)$ emanating from x_{i+1}^- between the extensions of the edges (p', x_{i+1}^-) and (q', x_{i+1}^-) beyond x_{i+1}^- . Any point inside \mathcal{W} can be connected via straight lines to all of p', q', x_{i+1}^- without any crossings. Consider a straight line s' through \mathcal{W} , not through any of x_{i+1}^-, p', q' , and crossing (p', q') . Now place ℓ_2 vertices along s' within \mathcal{W} , and connect all of them to all of x_{i+1}^-, p', q' via straight lines. All generated crossings are among the added edges. The drawing is still straight-line and good, and the number of added crossings is $\binom{\ell_2}{2}$. The outer face of the resulting drawing is again a triangle with two corners being p' and q' and the third corner being a vertex that was added on s' . We assign this latter vertex the role of the emerging vertex x_i^+ ; the other inserted vertices are the necessary singletons. With this, the invariant holds since $T(C_{i+1}) = T(C_i) \cup \{x_i^+\} \setminus \{x_{i+1}^-\}$.

This finishes the description of the drawing algorithm. We claim that the final drawing has the minimum possible number of crossings: We first give an upper bound on the number of crossings that we achieve, and then show that any drawing requires this number.

► **Lemma 3.** *The above algorithm produces at most $\sum_{i=1}^{\kappa} \lfloor \frac{1}{2}(n(C_i) - 3) \rfloor \lfloor \frac{1}{2}(n(C_i) - 4) \rfloor$ crossings.*

Proof. The algorithm started with a planar drawing of K_4 . We argued above that the i -th iteration (drawing C_i , which contains ℓ singletons) added

$$\frac{1}{2}\ell_1(\ell_1 + 1) + \frac{1}{2}\ell_2(\ell_2 - 1) = \lfloor \frac{1}{2}(\ell + 1) \rfloor \lfloor \frac{1}{2}(\ell + 2) \rfloor$$

crossings, where $\ell_1 = \lfloor (\ell + 1)/2 \rfloor$ and $\ell_2 = \lceil (\ell + 1)/2 \rceil$. Finally, observe that $\ell = n(C_i) - 5$ since all vertices of C_i except $T(C_i) \cup \{x_i^+, x_i^-\}$ are singletons. ◀

► **Lemma 4.** *Any good drawing of G requires at least $\sum_{i=1}^{\kappa} \lfloor \frac{1}{2}(n(C_i) - 3) \rfloor \lfloor \frac{1}{2}(n(C_i) - 4) \rfloor$ crossings.*

Proof. From Observation 1 we know that each cluster C_i contains a biclique $B(C_i) := K_{3, n(C_i)-3}$. By Zarankiewicz' formula, $K_{3,m}$ needs $\lfloor m/2 \rfloor \lfloor (m-1)/2 \rfloor$ crossings in any drawing. Thus, within each cluster we only introduce the optimal number of crossings.

However, we must argue that it is impossible for one crossing to belong to two or more clusters in an optimal drawing. This holds because nearly all of $V(C_i)$ does not belong to other clusters. More precisely, assume some other cluster C_j shares vertices with C_i ; we may assume $j < i$. Then all common vertices must appear in the first bag $X = T(C_i) \cup \{x_i^-\}$ of C_i . However, only three edges of those induced by X are in $B(C_i)$, and all three of them are incident to x_i^- . Since adjacent edges do not cross in a good drawing, no crossing can be shared between $B(C_i)$ and $B(C_j)$. ◀

► **Theorem 5.** *There is a linear time algorithm to compute the exact crossing number $cr(G)$ of any maximal pathwidth-3 graph G . Furthermore, $cr(G) = \overline{cr}(G)$, and the algorithm gives rise to a straight-line drawing where the anchor edges are not crossed.*

Proof. Optimality follows from Lemmas 3 and 4. The second part of the claim follows from the first and third invariant in the above algorithmic description. It remains to argue linear running time. Computing a path decomposition of width 3 (if it exists) can be done in linear

time $[1, 2]$. This path decomposition can be turned into an alternating path decomposition in linear time as well. On it we compute $cr(G)$ as the sum in Lemma 3 in linear time. ◀

Assume we are interested in the drawing achieving this solution. The drawing algorithm uses $O(n)$ operations, but this does not immediately imply linear time, since coordinates may become very small. We also cannot list all crossings, as there can be $\Theta(n^2)$ many. If, however, we are careful about how to place anchor-triplets, then singletons can be inserted while keeping all vertices at grid-points of an $O(n) \times O(n)$ -grid, and thus we require only linear time to compute and output the drawing. Details are given in Appendix B. We summarize:

► **Theorem 6.** *Every maximal pathwidth-3 graph on n vertices has a crossing-minimum drawing that is good, straight-line, and lies on a $28n \times 29n$ -grid. It can be found in $O(n)$ time.*

4 Approximation Algorithm for Pathwidth-3 Graphs

We now give an algorithm that draws graphs of pathwidth 3 (not necessarily maximal) such that the number of crossings is within a factor of 2 of the optimum. Roughly speaking, if the graph is 3-connected (technically, we will define a slightly weaker assumption *3-traceable*), then the algorithm for maximal pathwidth-3 graphs is applied, and the number of crossings is within a factor of 2. If the graph is not 3-traceable, then it can be split and the arising subdrawings can be “glued” together without increasing the approximation ratio.

4.1 3-traceable graphs

We first analyze graphs that satisfy a condition that is weaker than 3-connectivity. Define a *non-anchor vertex* to be a vertex that occurs in exactly one bag. Those are exactly v_1 , v_n , and all the singletons defined earlier.

► **Definition 7** (3-traceable graph). A graph G with an alternating path decomposition \mathcal{P} of width 3 is *3-traceable* if every non-anchor vertex has degree at least 3, and for all $1 \leq i \leq \kappa$, edge (x_{i-1}^+, x_i^-) exists.

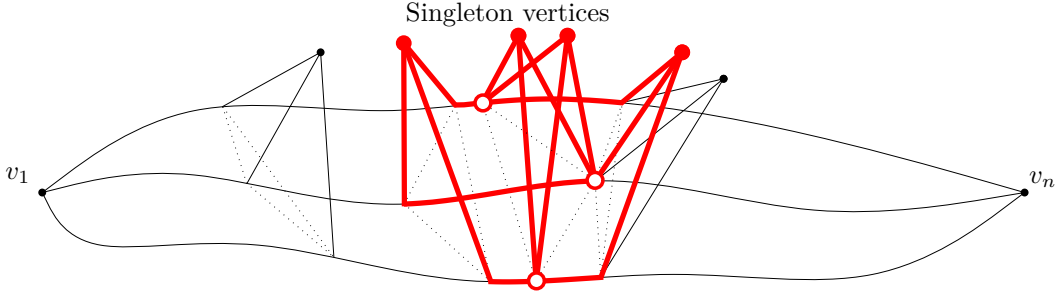
Assume we are given a 3-traceable graph G with an alternating path decomposition \mathcal{P} of width 3. We can first maximize G (obtaining G') by adding all edges that have both ends in one bag, but are not in G' yet. We then apply the algorithm described in Section 3 to G' , and finally delete the temporarily added edges again. We will show:

► **Lemma 8.** *Let G be a 3-traceable graph. Then the algorithm of Theorem 5 gives a drawing of G with at most $2cr(G)$ crossings.*

We first give a sketch of the proof. The main challenge is that a cluster C now does not necessarily contain a biclique $K_{3,n(C)-3}$. However, we can argue that G contains a subdivision of $K_{3,n(C)-3}$ that uses mostly vertices of C , but “borrows” a non-anchor vertex each from the nearest preceding and succeeding cluster that has such vertices. This subdivided $K_{3,n(C)-3}$ requires $cr(K_{3,n(C)-3})$ crossings. The main work is then in arguing that these subdivided bicliques cannot overlap much, or more precisely, that any crossing can belong to at most 2 of them. Lemma 8 then follows by applying the upper bound given in Lemma 3.

As before, let C_1, \dots, C_κ be the clusters of G with anchor-triplets $T(C_1), \dots, T(C_\kappa)$, and recall that we have an age-order $\{v_1, \dots, v_n\}$.

There are three types of edges in G . Type I are edges that are incident to non-anchor vertices. Type II are edges that have the form (x_{i-1}^+, x_i^-) for some $2 \leq i \leq \kappa$. Finally, Type III are the remaining edges (they connect vertices of some anchor-triplet $T(C_i)$, $1 \leq i \leq \kappa$).



■ **Figure 4** The structure of a 3-traceable graph. Dotted triangles mark anchor-triples with at least one adjacent singleton. In bold, we show one cluster biclique: the anchor vertices depicted as circles form one partition side. The left- and right-most bold singleton is “borrowed” from the preceding and succeeding singleton-containing cluster, respectively.

► **Observation 9.** Consider a 3-traceable graph. For any $1 \leq i < j \leq \kappa$, there are three vertex-disjoint paths $\Pi_{i,j}$ from $T(C_i)$ to $T(C_j)$ consisting exactly of the Type II edges (x_{k-1}^+, x_k^-) for $i < k \leq j$. Every non-anchor vertex attaches to the three different paths $\Pi := \Pi_{1,\kappa}$.

Proof. For any $1 \leq i < \kappa$, we have $T(C_{i+1}) = T(C_i) \cup \{x_i^+\} \setminus \{x_{i+1}^-\}$. By 3-traceability of G , edge (x_{i+1}^-, x_i^+) exists and $\Pi_{i,i+1}$ consists of two paths of length 0 (the common vertices of the triplets) and the third path being this edge. We obtain arbitrary $\Pi_{i,j}$ by extending $\Pi_{i,i+1}$ via $\Pi_{i+1,j}$. Since G is 3-traceable, the non-anchor vertices have degree 3 and are adjacent to the vertices of the anchor-triplet of their unique cluster; those lie on distinct paths of Π . ◀

This shows that G has $K_{3,n'}$ as a minor, where n' is the number of non-anchor vertices. Unfortunately this is not sufficient for crossing number arguments as contracting edges may increase the crossing number. Instead, we will use the above structure to extract a subdivision of $K_{3,n(C)-3}$ for each cluster C in such a way that these bicliques do not overlap “much.”

► **Definition 10.** Let C_i , $1 \leq i \leq \kappa$, be a cluster with at least one singleton. The *cluster biclique* of C_i , denoted $\mathcal{B}(C_i)$, is a subdivision of $K_{3,n(C_i)-3}$ obtained as follows, cf. Fig. 4:

- (a) The 3-side is formed by the three vertices of $T(C_i)$.
- (b) Every singleton w that belongs to C_i (there are $n(C_i) - 5$ of them) is one of the vertices on the side that will have $n(C_i) - 3$ vertices. We know that $\deg(w) = 3$ by 3-traceability, and it is adjacent to all of $T(C_i)$ as required for the biclique.
- (c) Let $i_- < i$ ($i_+ > i$) be maximal (minimal) such that cluster C_{i_-} (C_{i_+} , respectively) has a non-anchor vertex; among its non-anchor vertices, let w_- (w_+) be the youngest (oldest, respectively). If $i = 1$, we simply set $w_- := v_1$; if $i = \kappa$, we set $w_+ := v_n$. By Observation 9, we can establish three disjoint paths from w_- and w_+ to $T(C_i)$. Hence, add w_- and w_+ to the “big” side of $\mathcal{B}(C_i)$. Observe that in either case, w_- and w_+ are distinct from the singletons of C_i and their paths to $T(C_i)$.

► **Lemma 11.** Let e_1, e_2 be two edges of G without common endpoint. There are at most two cluster bicliques that contain both e_1 and e_2 .

Proof. We are done if at least one of e_1 and e_2 is of Type III, because then it belongs to no cluster biclique at all. Assume that one of e_1 and e_2 is of Type II, say $e_1 = (x_{i-1}^+, x_i^-)$ for some $2 \leq i \leq \kappa$. Edge e_1 may be used only for the cluster bicliques $\mathcal{B}(C_{j-})$ and $\mathcal{B}(C_{j+})$ where

$j^- < i$ ($j^+ \geq i$) is the maximal (minimal) index such that cluster C_{j^-} (C_{j^+} , respectively) has singletons. The fact that e_1 belongs to at most two cluster bicliques proves the claim.

Finally, assume that both e_1 and e_2 are of Type I, i.e., incident to distinct non-anchor vertices, say $y_1 \in C_i$ and $y_2 \in C_{i'}$. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the ordered subsequence of clusters that have at least one non-anchor vertex. A non-anchor vertex x can belong to at most three cluster bicliques, refer to Definition 10: the one of its “own” cluster $C \in \mathcal{C}'$, and those of the directly preceding and succeeding cluster in \mathcal{C}' . Assume that y_1 and y_2 are in three cluster bicliques. If $i = i'$, y_1 and y_2 are singletons of different age in C_i , and the two clusters directly preceding and succeeding C_i would have chosen distinct singletons of C_i , a contradiction. If $i \neq i'$, any overlap of three-element subsequences of \mathcal{C}' with distinct middle clusters has size at most 2, a contradiction. \blacktriangleleft

Proof of Lemma 8. We know from Lemma 3 that the algorithm of Theorem 5 gives a drawing with at most $\sum_{C \in \mathcal{C}} \lfloor \frac{1}{2}(n(C) - 3) \rfloor \lfloor \frac{1}{2}(n(C) - 4) \rfloor$ crossings. We need to consider only clusters C that have at least one singleton; for any other cluster we have $n(C) = 5$ and therefore its summand is 0. For any cluster C that has a singleton, we have $\mathcal{B}(C)$, a subdivision of $K_{3, n(C)-3}$, which requires at least $\lfloor \frac{1}{2}(n(C) - 3) \rfloor \lfloor \frac{1}{2}(n(C) - 4) \rfloor$ crossings in any good drawing \mathcal{D} of G . Any crossing in \mathcal{D} is created by two edges without common endpoints, and by Lemma 11, any such pair belongs to at most two cluster bicliques. Hence any drawing of G has at least

$$\frac{1}{2} \sum_{C \in \mathcal{C}} \lfloor \frac{1}{2}(n(C) - 3) \rfloor \lfloor \frac{1}{2}(n(C) - 4) \rfloor$$

crossings, yielding the 2-approximation. \blacktriangleleft

4.2 General pathwidth-3 graphs

A pair of vertices $\{u, v\}$ of a 2-connected graph G is called a *separation pair* if $G - \{u, v\}$ is not connected. Assume that the pathwidth-3 graph G is 2-connected but not 3-traceable. We will show that we can split the graph at separation pairs within anchor-triplets, draw the cut-components recursively, and merge them without introducing additional crossings. We start with a more general auxiliary statement.

► **Lemma 12.** *Let G be a 2-connected graph with a separation pair $\{u, v\}$. Consider a partition of G into two edge-disjoint connected subgraphs H_1, H_2 with $H_1 \cap H_2 = \{u, v\}$. Define $H_i^+ = H_i \cup \{(u, v)\}$ for $i = 1, 2$. Then $cr(H_1^+) + cr(H_2^+) \leq cr(G)$.*

Proof. Let \mathcal{D} be a drawing achieving $cr(G)$, and let \mathcal{D}_i be the subdrawing of \mathcal{D} corresponding to H_i . Each of the latter gives rise to a planarly embedded graph L_i of H_i , where crossings in \mathcal{D}_i are substituted by degree-4 vertices. We call edges in L_i *subedges*. We call a u - v -path in L_i an i -*path*, and for each $i = 1, 2$, we choose an i -path P_i . Let $\mathcal{D}_i^+ \supset \mathcal{D}$ be a drawing of H_i^+ where (u, v) is drawn into \mathcal{D}_i (without (u, v) if it already existed) following the route of P_{3-i} ; we have $cr(H_i^+) \leq cr(\mathcal{D}_i^+)$. Clearly, any crossing in any \mathcal{D}_i^+ has a counterpart in \mathcal{D} . Inversely, any crossing in \mathcal{D} can show up in at most one of $\mathcal{D}_1^+, \mathcal{D}_2^+$, except for crossings between edges of P_1 and P_2 —so-called *path-crossings*. We show that for each crossing that we count in both \mathcal{D}_1^+ and \mathcal{D}_2^+ , there is at least another crossing in \mathcal{D} that is in neither $\mathcal{D}_1^+, \mathcal{D}_2^+$.

We can assume that any choice of P_1, P_2 gives path-crossings, as otherwise we would be done. Furthermore, for $i = 1, 2$, we can assume there are no two subedge-disjoint i -paths; otherwise, we can pick the one with fewer paths-crossings with P_{3-i} as P_i and be done. Similarly, we can account for crossings on a subpath $P'_i = (w \rightarrow w') \subseteq P_i$ if there is another

subedge-disjoint subpath connecting w to w' in L_i . Let $F_i \subset P_i$, $i = 1, 2$, be the subedges that are in every i -path. We only have to account for crossings between F_1 and F_2 .

Assume there is a path-crossing (e_1, e_2) , $e_i \in F_i$, even though we choose a crossing minimal insertion route for P_1 in L_2 . Therefore the ends of e_1 lie in different faces of the planar graph L_2 . In consequence e_2 lies on a cycle $Q \in L_2$ separating the ends of e_1 from each other. Let P'_2 and P''_2 be the subpaths after deleting e_2 from P_2 . The subgraph $S = P'_2 \cup (Q \setminus \{e_2\}) \cup P''_2$ connects u to v in L_2 even though $e_2 \notin S$ —a contradiction to $e_2 \in F_2$. \blacktriangleleft

We will draw cut-components inside triangles bounded by their three oldest vertices.

► **Lemma 13.** *Let G be a 2-connected graph with an alternating path decomposition \mathcal{P} of width 3. Then there exists an algorithm to create a straight-line drawing of G with at most $2cr(G)$ crossings. All anchor-edges are drawn without crossings, and the three oldest vertices $\{v_1, v_2, v_3\}$ form the corners of the triangular convex hull of the drawing.*

Proof. We prove the result by induction on the size of the graph.

Base case: G is 3-traceable or a K_4 . If $G = K_4$, the claim is obvious. Otherwise, we apply Lemma 8. However, the algorithm of Theorem 5 used therein grows the drawing “outwards”, while we would now like the oldest vertices to form the outer triangle. Thus we apply the algorithm for the reverse path decomposition; this makes (by suitably placing the last vertex) $T(C_1) = \{v_1, v_2, v_3\}$ the outer face and draws it as a triangle.

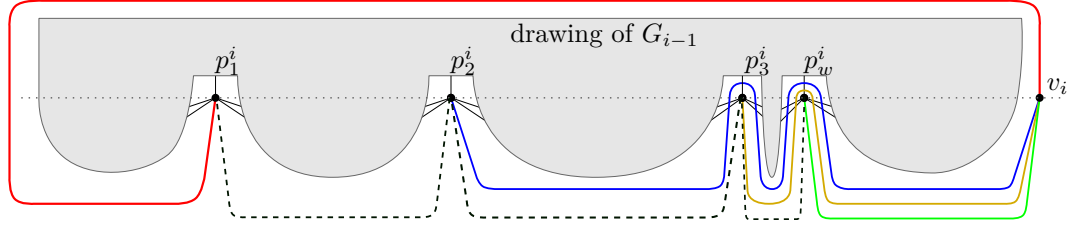
Induction Step: G is neither 3-traceable nor a K_4 . For every non-anchor vertex $w \neq v_1$ of degree 2, let p_w, q_w be its adjacent anchor vertices. We can temporarily remove w from G , ensure that the reduced graph contains edge (p_w, q_w) , draw the reduced graph, and—since (p_w, q_w) will be drawn crossing free by the induction hypothesis—re-insert each w with $(p_w, w), (w, q_w)$ crossing-free close to the drawing of (p_w, q_w) . Similarly, we can remove v_1 if it has degree 2: We can choose an age-order of the reduced graph G' such that the neighbours of v_1 are among the three oldest vertices of G' and hence draw G' such that the neighbours of v_1 are on the outer-triangle; then v_1 can be re-inserted on the outside to form the desired outer triangle. If the graph became 3-traceable by these operations, we are done (base case). Otherwise, we can now assume that all non-anchor vertices have degree 3.

Since G is not 3-traceable, $(x_{i-1}^+, x_i^-) \notin G$ for some $2 \leq i \leq \kappa$. There exists a unique bag X_j , the common bag of C_{i-1} and C_i , that contains both x_{i-1}^+ and x_i^- . Let p, q be the two other vertices in this bag, and observe that $T(C_{i-1}) = \{p, q, x_i^-\}$ while $T(C_i) = \{p, q, x_{i-1}^+\}$. Let G_ℓ be the graph induced by all vertices that appear in bags $\mathcal{P}_\ell := [X_1, X_{j-2}]$, and let G_r be the graph induced by all vertices that appear in bags $\mathcal{P}_r := [X_{j+2}, X_\kappa]$. Any edge of G appears in G_ℓ or G_r , since $\{x_i^-, x_{i-1}^+\}$ is the only vertex-pair that existed in bags of \mathcal{P} , but neither of \mathcal{P}_ℓ nor \mathcal{P}_r . Clearly, $\{p, q\}$ is a separation pair with $G_\ell \cap G_r = \{p, q\}$.

Define $G_\ell^+ = G_\ell \cup \{(p, q)\}$ and $G_r^+ = G_r \cup \{(p, q)\}$. By the addition of edge (p, q) (if it did not already exist), both graphs are 2-connected. Apply induction to G_r^+ (with path decomposition \mathcal{P}_r) and G_ℓ^+ (with the path decomposition \mathcal{P}_ℓ). Since p, q belong to the first bag of \mathcal{P}_r , we can ensure that they are among the three oldest vertices of G_r^+ . We obtain two drawings $\mathcal{D}_1^+, \mathcal{D}_2^+$ in both of which (p, q) is not crossed. We can insert (affinely transformed) \mathcal{D}_2^+ , which has (p, q) on its bounding triangle, along (p, q) in \mathcal{D}_1^+ without additional crossings. Finally, we remove edge (p, q) from the resulting drawing if $(p, q) \notin E(G)$.

By induction hypothesis, $cr(\mathcal{D}_\ell^+) \leq 2cr(G_\ell^+)$ and $cr(\mathcal{D}_r^+) \leq 2cr(G_r^+)$. By Lemma 12, $cr(G_\ell^+) + cr(G_r^+) \leq cr(G)$ and since the gluing gave no new crossings, the claim follows. \blacktriangleleft

We are now ready to establish the theorem for general pathwidth-3 graphs.



■ **Figure 5** The construction for higher pathwidth: edge routings when adding vertex v_i .

► **Theorem 14.** *Let G be any pathwidth-3 graph. We have $\overline{cr}(G) \leq 2cr(G)$, and a linear time algorithm to create a good straight-line drawing of G with at most $2cr(G)$ crossings.*

Proof. (Sketch) If G is 2-connected, then the result holds by Lemma 13. It is well known that $cr(G)$ is additive over the 2-connected components of G . When gluing at cut-vertices, one needs to ensure that the cut-vertex is on the outer face of the drawing to be inserted into the other. We can achieve this while maintaining a straight-line drawing by choosing appropriate path decompositions; see Appendix C for details. The running time follows from the arguments for Theorem 5. ◀

5 Approximation Algorithm for Graphs of Higher Pathwidth

We now study the crossing number of graphs that have pathwidth $\mathbf{w} \geq 4$, and are maximal within this class. We give an algorithm to draw such graphs, and show that the number of crossings in the resulting drawing is within a factor of $4\mathbf{w}^3$ of the crossing number. As opposed to Section 3, the drawings we create here are not straight-line drawings.

As before we assume that we have an alternating path decomposition $\mathcal{P} = \{X_i\}_{1 \leq i \leq \xi}$ of width \mathbf{w} . We again use the *age-order* $\{v_1, \dots, v_n\}$ of the vertices of G . Define G_i to be the graph induced by vertices v_1, \dots, v_i , and use $\deg_{G_i}(v)$ to denote the number of neighbors that v has within graph G_i . For any $1 \leq i \leq n$, let the *predecessors* of vertex v_i be those neighbors that are older. We will only use this concept for $i \geq \mathbf{w} + 1$, which implies that v_i has exactly \mathbf{w} predecessors by maximality of G . We enumerate them as $\{p_1^i, \dots, p_{\mathbf{w}}^i\}$ in age-order, with p_1^i the oldest.

5.1 Drawing algorithm

We create a drawing of G by starting with $G_{\mathbf{w}+1}$ (the graph induced by $v_1, \dots, v_{\mathbf{w}+1}$) and then iteratively adding vertex v_i . We maintain the following invariants for the drawing of G_i :

- Vertex v_j is drawn at $(j, 0)$ for all $1 \leq j \leq i$.
- The drawing is contained in the half-space $\{(x, y) : x \leq i\}$.
- All vertices w in the bag introducing v_i are *bottom-visible*, i.e., the vertical ray downward from w does not intersect any edge.

We start by placing $v_1, \dots, v_{\mathbf{w}+1}$ at their specified coordinates, and draw the edges between them as half-circles above the x -axis. This satisfies the above invariants and gives rise to $\binom{\mathbf{w}+1}{4}$ crossings: any four vertices create a K_4 that has one crossing, and any crossing belongs to exactly one such K_4 .

Assume G_{i-1} is drawn and consider v_i , for $i \geq \mathbf{w} + 2$. Place v_i as specified, i.e., to the right of all previous vertices and edges. Let $p_1^i, \dots, p_{\mathbf{w}}^i$ be the predecessors of v_i , all of which

are bottom-visible by the invariant. We draw the edges to them using two different methods (and then redraw previous edges as a third step for each i):

- The edge to p_1^i (the oldest predecessor) is routed counterclockwise around the drawing of G_{i-1} until it is below but slightly to the left of p_1^i , from where it connects to p_1^i . We need no crossings, and all predecessors remain bottom-visible.
- All other $\mathbf{w} - 1$ edges incident to v_i are routed together as a bundle from v_i leftward below the drawing of G_{i-1} . This allows v_i to be bottom-visible. Whenever the bundle is slightly to the right of some p_k^i , $\mathbf{w} \geq k \geq 2$, one of the bundle's lines (the lowest one) connects to p_k^i . The remaining bundle lines go counterclockwise around p_k^i , in its direct vicinity, until they are to the left of p_k^i and below G_{i-1} . The bundle hence crosses every edge incident to p_k^i in G_{i-1} , but no other edges, and p_k^i remains bottom-visible. This drawing scheme continues until the last bundle line connects to p_2^i .
- Finally, we redraw the edges (p_{k-1}^i, p_k^i) for $3 \leq k \leq \mathbf{w}$; they exist by maximality. Since both ends of any such edge are bottom-visible, we can redraw it without crossing by staying below the entire drawing, including the newly drawn edges from v_i . We remove the previous drawings of these edges and retain bottom-visibility of the vertices in the current bag.

5.2 Upper-bounding the number of crossings

With the routing as described, some edges cross twice for $\mathbf{w} \geq 5$ (e.g., edge (p_2^i, v_i) crosses edge (p_3^i, p_5^i) both near p_3^i and near p_5^i). We can avoid such crossings by local re-drawings, which can only improve the overall number of crossings. But in our counting of crossings we will not take advantage of this.

We want to bound the number of crossings incurred when drawing vertex v_i , $i \geq \mathbf{w} + 2$. No new crossings occur in the vicinity of p_1^i or p_2^i . Consider the routing of edge (p_j^i, v_i) in the vicinity of p_k^i for some $3 \leq j < k \leq \mathbf{w}$. This edge crosses any edge incident to p_k^i with two exceptions: It does not cross (p_k^i, v_j) , since we ordered edges within the bundle appropriately. And it does not cross the edge (p_{k-1}^i, p_k^i) , since we re-routed that edge to be without crossings after the introduction of v_i . Therefore edge (p_j^i, v_i) crosses at most $\deg_{G_i}(p_k^i) - 2$ other edges in the vicinity of p_k^i . Summing up over all k and over the $\mathbf{w} - 1$ edges added within the bundle of v_i gives:

► **Observation 15.** *Drawing vertex v_i gives at most $\sum_{j=3}^{\mathbf{w}} (j-2)(\deg_{G_i}(p_j^i) - 2)$ new crossings.*

To simplify this bound, we upper-bound the degrees.

► **Observation 16.** *For all $k \geq 4$, $\deg_{G_i}(p_3^i) \geq \deg_{G_i}(p_k^i)$. Thus, drawing vertex v_i adds at most $\frac{(\mathbf{w}-1)(\mathbf{w}-2)}{2}(\deg_{G_i}(p_3^i) - 2)$ new crossings.*

Proof. Vertices p_k^i and p_3^i are adjacent. Besides this, any predecessor u of p_k^i is a predecessor of p_3^i , or it was introduced after p_3^i . In both cases, u is adjacent to p_3^i as well. Since we are looking at G_i (and not full G), any vertex so far introduced after p_k^i is adjacent to both p_k^i and p_3^i . This proves the first part of the claim and the second follows from Observation 15. ◀

Define again (and compatible to before) an *anchor-triplet* T to be three vertices that are the oldest vertices of some bag $X \neq X_1$. Note that, again, T forms a triangle by maximality. Also, T again defines a *cluster* consisting of all bags that contain all of T . Clearly, the bags of a cluster are again consecutive. However, in contrast to before, clusters may overlap in more than one bag. Figure 8 in the appendix gives an example.

We say a vertex u is *introduced by cluster C* if u appears in C , but not in $G_{\mathbf{w}+1}$ or in any cluster that ends at an earlier bag. (This is quite similar to the concept of singletons used earlier, except that a vertex that belongs to only one bag may now belong to multiple clusters, and is considered to be introduced only by the cluster that ends earliest.) Let $i(C)$ be the number of vertices introduced by a cluster C .

► **Observation 17.** *Let C be a cluster with $T(C) = \{p_1, p_2, p_3\}$ in age-order. Then the first bag of C introduces p_3 , $i(C) \leq n(C) - (\mathbf{w} + 1)$, and for any vertex v_i introduced by C we have $\deg_{G_i}(p_3) \leq n(C) - 1$.*

Proof. Vertex p_3 is adjacent to $\{p_1, p_2\}$ and so the bag X introducing p_3 contains $T(C)$. But no earlier bag contains p_3 , so X is the first bag of C . Any vertex in X appears in some earlier cluster (or in $G_{\mathbf{w}+1}$) and so was not introduced by C . Finally G_i considers only bags of C or earlier clusters, and so any neighbour of p_3 in G_i belongs to C . ◀

We can now restate the number of crossings achieved as follows:

► **Lemma 18.** *The above drawing algorithm for a maximal graph of pathwidth $\mathbf{w} \geq 4$ produces at most the following number of crossings:*

$$\binom{\mathbf{w}+1}{4} + \sum_{C \in \mathcal{C}} 2(\mathbf{w}-1)(\mathbf{w}-2) \left\lfloor \frac{n(C)-3}{2} \right\rfloor \left\lfloor \frac{n(C)-4}{2} \right\rfloor.$$

Proof. Graph $G_{\mathbf{w}+1}$ contributes $\binom{\mathbf{w}+1}{4}$ crossings. Each vertex v_i introduced by some cluster C adds at most $\frac{(\mathbf{w}-1)(\mathbf{w}-2)}{2}(\deg_{G_i}(p_3) - 2)$ crossings from Observation 16; observe that p_3^i is the youngest vertex $p_3 \in T(C)$. Applying Corollary 17 and summing over the $i(C) \leq n(C) - 5$ vertices introduced by C (Observation 17 and $\mathbf{w} \geq 4$), the number of crossings added by C is at most

$$\frac{(\mathbf{w}-1)(\mathbf{w}-2)}{2}(n(C)-3)(n(C)-5) \leq 2(\mathbf{w}-1)(\mathbf{w}-2) \left\lfloor \frac{n(C)-3}{2} \right\rfloor \left\lfloor \frac{n(C)-4}{2} \right\rfloor.$$

◀

5.3 Lower-bounding the crossing number

We know that our initial graph $G_{\mathbf{w}+1} = K_{\mathbf{w}+1}$ requires at least $\Theta(\mathbf{w}^4)$ crossings, see [11] for the currently best bounds. For us, the rather trivial $cr(G_{\mathbf{w}+1}) \geq \frac{1}{5}\binom{\mathbf{w}+1}{4}$ will suffice.

Every cluster C contains $B(C) := K_{3, n(C)-3}$, its *cluster biclique* with $T(C)$ as one partition set, and thus needs at least $\left\lfloor \frac{n(C)-3}{2} \right\rfloor \left\lfloor \frac{n(C)-4}{2} \right\rfloor$ crossings in any drawing by Zarankiewicz' formula. However, any one crossing may belong to multiple cluster bicliques, and so may be counted repeatedly.

► **Lemma 19.** *Consider a good drawing of a maximal graph G of pathwidth $\mathbf{w} \geq 4$. Any crossing belongs to at most $\mu = 2\mathbf{w} - 5$ cluster-bicliques.*

Sketch of Proof. We only prove $\mu \in O(\mathbf{w})$ here; a full proof is in Appendix D. Let χ be the vertices involved in a crossing. Let X_i be the last bag where one of χ is introduced and X_k be the first bag where one of χ is forgotten. For any cluster C whose biclique uses the crossing, each vertex of χ must appear in some bag of C . From this one can derive that X_i or X_{k-1} must belong to C . Since any bag belongs to $O(\mathbf{w})$ clusters, the bound follows. ◀

► **Corollary 20.** *Any good drawing of G has at least the following number of crossings:*

$$\frac{1}{\mu+1} \left(\frac{1}{5} \binom{\mathbf{w}+1}{4} + \sum_{C \in \mathcal{C}} \left\lfloor \frac{n(C)-3}{2} \right\rfloor \left\lfloor \frac{n(C)-4}{2} \right\rfloor \right)$$

Proof. Graph $G_{\mathbf{w}+1}$ needs at least $\frac{1}{5} \binom{\mathbf{w}+1}{4}$ crossings. Any cluster biclique $B(C)$ needs at least $\left\lfloor \frac{n(C)-3}{2} \right\rfloor \left\lfloor \frac{n(C)-4}{2} \right\rfloor$ crossings. In any drawing of G , crossings are counted in at most μ bicliques, and also in $G_{\mathbf{w}+1}$. ◀

Combining the upper and lower bound immediately gives the main result of this section.

► **Theorem 21.** *Let G be a maximal graph of pathwidth $\mathbf{w} \geq 4$. The described algorithm runs in linear time and finds a drawing of G with at most $2(\mathbf{w}-1)(\mathbf{w}-2)(\mu+1)cr(G) \leq 4\mathbf{w}^3 cr(G)$ crossings. In particular, for any constant pathwidth \mathbf{w} , we have an $O(1)$ -approximation of the crossing number. The drawing is poly-line on a $4n \times \mathbf{w}n$ grid.*

Proof. The approximation ratio comes from combining the upper bound of Lemma 18 with the lower bound of Corollary 20, and the observation that $5 < 2(\mathbf{w}-1)(\mathbf{w}-2)$ for $\mathbf{w} \geq 4$. The runtime for the decomposition has already been argued for Theorem 5; all the remaining algorithmic steps can be done in linear time as well.

It remains to argue the complexity of the grid. For each vertex, we add one extra (vertex-free) column just before and one just after it. Whenever we need to route “around” some vertex p_j^i , we use its three columns to place all necessary bends (cf. Fig. 5). Furthermore, we use one additional column for each edge from v_i to its oldest predecessor p_1^i . Therefore, we need no more than $4n$ columns for all vertices and bends.

Now subdivide each edge with a dummy-node whenever it crosses a column without having a bend- or endpoint there. What results is a so-called hierarchical drawing (turned sideways). We can rearrange this easily, column by column, so that the height of the drawing is dominated by the column with the maximum number of vertices, bends, or dummy-nodes. Any of the columns used for routings to first predecessors is crossed by at most n edges, each edge crossing twice or having two bends. Thus these columns require a height of at most $2n$. Any of the other columns could be crossed by almost all edges, but all edges are routed x -monotonically within there, and hence cross any column at most once. A graph of pathwidth \mathbf{w} has at most $\mathbf{w}n$ edges, and so the bound follows. ◀

6 Conclusions and Open Questions

We have shown that the path decomposition of a graph can be used to efficiently compute or bound the crossing number of a graph. As such, it is the first successful use of such graph decomposition in the context of crossing numbers (besides the use of a tree decomposition in the special case that $cr(G)$ is bounded by a constant [13, 16]). There are, of course, several interesting questions remaining:

- Can we attain stronger approximation results for general pathwidth-3 graphs? The proven ratio of 2 may simply be due to a too weak lower bound, and we, in fact, do currently not know an instance where the algorithm does not obtain the optimum (possibly up to some lower order terms).
- Can we approximate $cr(G)$ for arbitrary (not maximal) pathwidth- \mathbf{w} -graphs?
- We only showed weak NP-completeness for the weighted crossing number version on pathwidth-restricted graphs, and it is natural to ask if this can be strengthened to unweighted graphs.

Finally, there is of course the question whether we can use the stronger tool of tree decompositions, instead of path decompositions, to achieve crossing number results.

References

- 1 H.L. Bodlaender. A linear-time algorithm for finding tree-decompositions of small treewidth. *SIAM J. Comput.*, 25(6):1305–1317, 1996.
- 2 H.L. Bodlaender and T. Kloks. Efficient and constructive algorithms for the pathwidth and treewidth of graphs. *J. Algorithms*, 21(2):358–402, 1996.
- 3 D. Bokal. On the crossing numbers of cartesian products with paths. *J. Comb. Theory Ser. B*, 97(3):381–384, May 2007.
- 4 S. Cabello. Hardness of approximation for crossing number. *Discrete & Computational Geometry*, 49(2):348–358, 2013.
- 5 S. Cabello and B. Mohar. Crossing number and weighted crossing number of near-planar graphs. *Algorithmica*, 60(3):484–504, 2011.
- 6 M. Chimani and P. Hliněný. A tighter insertion-based approximation of the crossing number. *Journal of Combinatorial Optimization*, pages 1–43, 2016.
- 7 M. Chimani and P. Hliněný. Inserting multiple edges into a planar graph. In *SoCG 2016*, pages 30:1–30:15. LIPIcs, 2016.
- 8 M. Chimani, P. Hliněný, and P. Mutzel. Vertex insertion approximates the crossing number for apex graphs. *European Journal of Combinatorics*, 33:326–335, 2012.
- 9 J. Chuzhoy. An algorithm for the graph crossing number problem. In *STOC '11*, pages 303–312. ACM, 2011.
- 10 B. Courcelle. The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. *Information and Computation*, 85(1):12–75, 1990.
- 11 E. de Klerk, J. Maharry, D.V. Pasechnik, R.B. Richter, and G. Salazar. Improved bounds for the crossing numbers of $K_{m,n}$ and K_n . *SIAM J. Discr. Math.*, 20(1):189–202, 2006.
- 12 I. Gitler, P. Hliněný, J. Leanos, and G. Salazar. The crossing number of a projective graph is quadratic in the face-width. *Electronic Journal of Combinatorics*, 15(1):#R46, 2008.
- 13 M. Grohe. Computing crossing numbers in quadratic time. *J. Comput. Syst. Sci.*, 68(2):285–302, 2004.
- 14 P. Hliněný and M. Chimani. Approximating the crossing number of graphs embeddable in any orientable surface. In *SODA '10*, pages 918–927, 2010.
- 15 P. Hliněný and G. Salazar. Approximating the crossing number of toroidal graphs. In *ISAAC '07*, LNCS 4835, pages 148–159. Springer, 2007.
- 16 K-I. Kawarabayashi and B. Reed. Computing crossing number in linear time. In *STOC '07*, pages 382–390, 2007.
- 17 D.J. Kleitman. The crossing number of $K_{5,n}$. *Journal of Combinatorial Theory*, 9(4):315–323, 1970.
- 18 M. Klešč and J. Petrillová. The crossing numbers of products of path with graphs of order six. *Discussiones Mathematicae Graph Theory*, 33(3):571–582, 2013.
- 19 T. Kloks. *Treewidth, Computations and Approximations*. LNCS 842. Springer, 1994.
- 20 S. Pan and R.B. Richter. The crossing number of K_{11} is 100. *Journal of Graph Theory*, 56(2):128–134, 2007.
- 21 R.B. Richter and G. Salazar. The crossing number of $P(N, 3)$. *Graphs and Combinatorics*, 18(2):381–394, 2002.
- 22 M. Schaefer. The graph crossing number and its variants: A survey. *Electronic Journal of Combinatorics*, #DS21, May 15, 2014.
- 23 I. Vrt'o. Crossing numbers of graphs: A bibliography. <ftp://ftp.ifi.savba.sk/pub/imrich/crobib.pdf>, 2014.

A

 NP-hardness: Details

Theorem 2 states the NP-hardness of weighted crossing number even for graphs of pathwidth 4; this section expands on the exposition of this.

The weighted rectilinear crossing number problem asks: Given a graph $G = (V, E)$, edge weights $w: E \rightarrow \mathbb{N}_0^+$, and a threshold K , is there a straight-line drawing \mathcal{D} of G such that

$$wcr(\mathcal{D}) := \sum_{\substack{e_1, e_2 \in E, \\ e_1 \text{ and } e_2 \text{ cross in } \mathcal{D}}} w(e_1) \cdot w(e_2) \leq K \quad ?$$

Our reduction is from PARTITION, defined as follows. Given n positive integers a_1, \dots, a_n with $\sum_{i=1}^n a_i = 2S$, does there exist a $J \subset \{1, \dots, n\}$ such that $\sum_{i \in J} a_i = S$. Given a PARTITION instance \mathcal{I} , define graph G as described in the proof sketch as a $2n+2$ -cycle Q and n chords $e_i = (x_i, y_i)$ with weight a_i for $i = 1, \dots, n$. We must show that \mathcal{I} is a yes-instance if and only if G has a straight-line drawing \mathcal{D} with $wcr(\mathcal{D}) \leq S^2 - c$, where $c = \frac{1}{2} \sum_{i=1}^n a_i^2$ depends only on \mathcal{I} .

Assume first that there exists some $J \subset \{1, \dots, n\}$ with $\sum_{i \in J} a_i = S$. Figure 2 shows how to create a straight-line drawing of G : Place vertices x_1, \dots, x_n on the left legs of an X -shape, and vertices y_1, \dots, y_n on the right legs of the X , using the upper/lower leg depending on whether $i \in J$. With the help of x_0 and y_0 , the cycle can then be completed without crossing.

Consider a pair i, j with $i \in J$ and $j \notin J$. Then e_i is drawn between the two upper legs of the X (hence inside Q) and while e_j is drawn between the two lower legs of the X (hence outside Q), which means that they cannot cross. Also no edge of Q has a crossing. In consequence, the number of crossings is at most

$$\begin{aligned} \sum_{i, j \in J} a_i \cdot a_j + \sum_{i, j \notin J} a_i \cdot a_j &= \frac{1}{2} \left(\left(\sum_{i \in J} a_i \right)^2 - \sum_{i \in J} a_i^2 \right) + \frac{1}{2} \left(\left(\sum_{i \notin J} a_i \right)^2 - \sum_{i \notin J} a_i^2 \right) \\ &= \frac{1}{2} \left(S^2 - \sum_{i \in J} a_i^2 \right) + \frac{1}{2} \left(S^2 - \sum_{i \notin J} a_i^2 \right) = S^2 - c \end{aligned}$$

as desired.

For the other direction, assume that we have a straight-line drawing \mathcal{D} of G with $wcr(\mathcal{D}) \leq S^2 - c$. Since $c > 0$, no edge of Q can have a crossing. Define J to be the indices of all those edges e_i that are drawn inside Q . Any two such edges must cross each other, since the order of their endpoints is interleaved on Q . Likewise, any two edges e_i, e_j with $i, j \notin J$ must cross each other. In consequence, we have

$$wcr(\mathcal{D}) \geq \sum_{i, j \in J} a_i \cdot a_j + \sum_{i, j \notin J} a_i \cdot a_j = \frac{1}{2} \left(\sum_{i \in J} a_i \right)^2 + \frac{1}{2} \left(\sum_{i \notin J} a_i \right)^2 - c$$

Define $d = \sum_{i \in J} a_i - S = S - \sum_{i \notin J} a_i$ (note that d could be positive or negative). Then

$$wcr(\mathcal{D}) \geq \frac{1}{2} (S - d)^2 + \frac{1}{2} (S + d)^2 - c = S^2 + d^2 - c.$$

But we assumed $wcr(\mathcal{D}) \leq S^2 - c$, which implies $d^2 = 0 = d$ and hence $\sum_{i \in J} a_i = S$ as desired.

B Proof of Theorem 6

We explain how to place points for the algorithm in Section 3 so that the resulting drawing has linear coordinates. This involves a paradigm-shift in explaining how the drawing is created. In Section 3, we added vertices from the point of view of adding cluster C_i . This added half of the singletons near (x_i^-, x_{i-1}^+) , and the other half near (x_{i+1}^-, x_i^+) . We now change this around, and describe the algorithm in terms of all those singletons (coming from both C_i and C_{i-1}) that need to be added near one edge (x_i^-, x_{i-1}^+) . Let there be s_i such singletons (in terms of the notation of Section 3, we have $s_i = \ell_2(C_{i-1}) - 1 + \ell_1(C_i)$).

We first explain how to place all anchor vertices and v_1, v_n . We first split the vertices except v_1 into three groups. We put v_2 in group \mathcal{G}_T (“top”), v_3 in \mathcal{G}_L (“(lower) left”), and v_4 in \mathcal{G}_R (“(lower) right”). For any edge (x_i^-, x_{i-1}^+) , $i \geq 2$, its incident vertices are in the same group. We now place the considered vertices as follows (see also Fig. 7)¹:

- v_1 is placed at the origin.
- v_2 is placed at $(0, 10n)$, i.e., on the vertical upward ray from v_1 .
- v_3 is placed at $(-10n, -10n)$, i.e., on the diagonal downward-left ray from v_1 .
- v_4 is placed at $(10n, -10n)$, i.e., on the diagonal downward-right ray from v_1 .
- Now, iteratively for $i = 2, \dots, \kappa$, consider edge (x_i^-, x_{i-1}^+) . Vertex x_i^- has already been placed, while we do not have a placement for x_{i-1}^+ yet.
 - If $x_i^- \in \mathcal{G}_T$, then place x_{i-1}^+ on the vertical ray upward from v_1 , and $s_i + 5$ units higher than x_i^- .
 - If $x_i^- \in \mathcal{G}_L$, then place x_{i-1}^+ on the diagonal downward-left ray from v_1 , and $s_i + 4$ units farther left of and $s_i + 4$ units further down from x_i^- .
 - If $x_i^- \in \mathcal{G}_R$, then place x_{i-1}^+ on the diagonal downward-right ray from v_1 , and $s_i + 4$ units farther right of and $s_i + 4$ units further down from x_i^- .

One immediately verifies that this placement gives a planar drawing of the graph induced by the so-far considered vertices: Any edge either lies on a ray or connects two different rays, and as we go along in age-order, the current anchor triangle always forms the outer-face and the next vertex is placed outside of it. We briefly analyze the size of this drawing:

► **Claim 22.** *The drawing uses only points in the range $(-14n, 14n) \times (-14n, 15n)$.*

Proof. Consider the topmost vertex above v_1 . In the worst case, all the vertices are placed above v_1 . Thus, the largest y -coordinate is at most $10n + (\kappa - 1)5 + (n - \kappa) < 15n$, where $n - \kappa$ is the upper bound on the number of all singletons. Similarly, any vertex on the other two rays has horizontal and vertical distance less than $10n + 4n$ from v_1 , and the claim follows. ◀

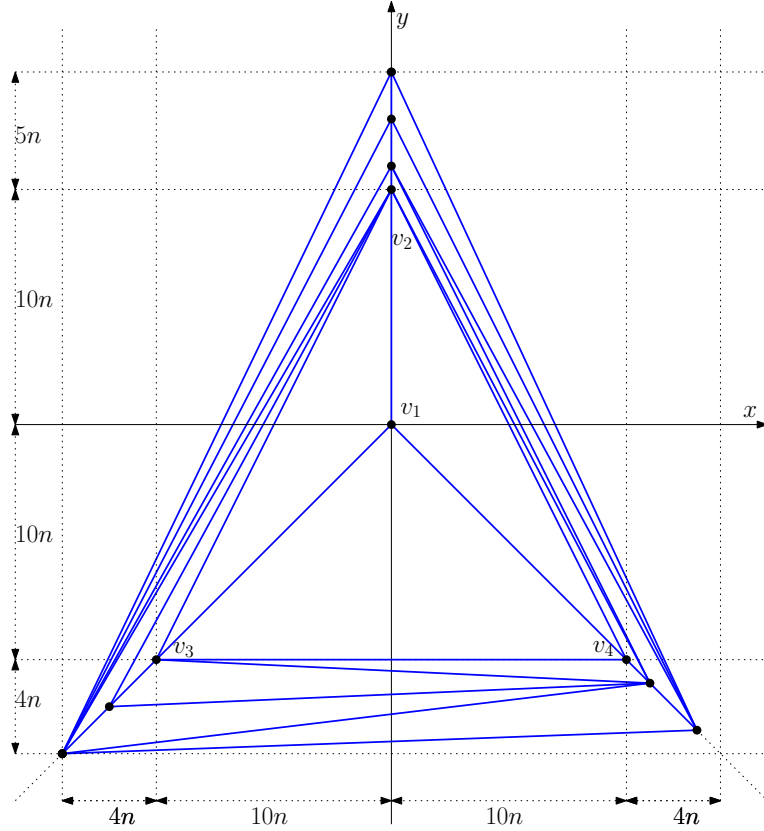
► **Claim 23.** *Any edge (u, v) from the left-down ray to the right-down ray has slope in $(-\frac{1}{5}, \frac{1}{5})$.*

Proof. We know that $x(u) = y(u) = -10n - k$ for some $0 \leq k < 4n$, and $y(v) = -x(v) = -10n - \ell$ for some $0 \leq \ell < 4n$. Assume $\ell \leq k$, i.e., the slope is non-negative (the other case is symmetric). The slope of the edge is hence

$$\frac{-10n - \ell - (-10n - k)}{10n + \ell - (-10n - k)} = \frac{k - \ell}{20n + \ell + k} < \frac{4n}{20n} = \frac{1}{5}.$$

◀

¹ The coordinates are chosen to be easy to define and analyze; the constant factor could likely be improved by making more careful choices.



■ **Figure 6** The overall layout (to scale).

► **Claim 24.** Any edge (u, v) from the left-down ray to the vertical-up ray has slope in $(\frac{10}{7}, 2.9)$.

Proof. We know that $x(u) = y(u) = -10n - k$ for some $0 \leq k < 4n$, $x(v) = 0$, and $y(v) = 10n + \ell$ for some $0 \leq \ell < 5n$. The slope of the edge is hence

$$\frac{10n + \ell - (-10n - k)}{-(-10n - k)} = \frac{20n + k + \ell}{10n + k}$$

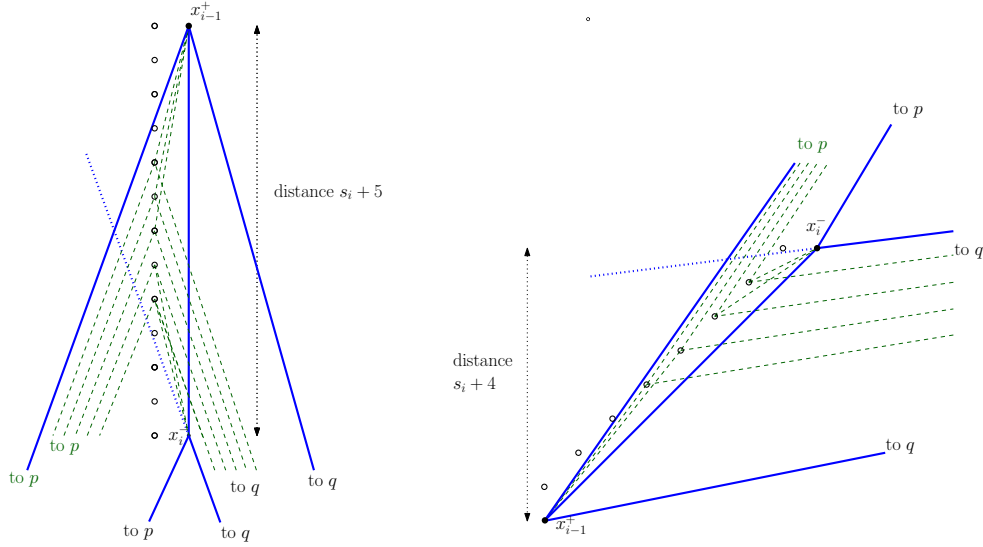
and we observe

$$\frac{10}{7} = \frac{20n}{14n} < \frac{20n + k + \ell}{10n + k} < \frac{29n}{10n} = 2.9.$$

◀

We must now add the points for singletons. Observe that any such vertex is placed “near” an edge (x_i^-, x_{i-1}^+) for some index $i \geq 2$, and is then connected either to all of $T(C_{i-1})$, or to all of $T(C_i)$. We must hence argue that near any edge (x_i^-, x_{i-1}^+) , we can find s_i grid points, each of which allows straight lines to all of $T(C_i) \cup T(C_{i-1})$ while intersecting only edge (x_i^-, x_{i-1}^+) . We distinguish cases depending on to which of the groups $\mathcal{G}_T, \mathcal{G}_L, \mathcal{G}_R$ the two vertices x_i^-, x_{i-1}^+ belong.

Case 1: $x_i^-, x_{i-1}^+ \in \mathcal{G}_T$: Let p, q be the two vertices in $T(C_i) \cap T(C_{i-1})$. They are not in \mathcal{G}_T , and on different rays, say $p \in \mathcal{G}_L$ and $q \in \mathcal{G}_R$. From a point x , we can see the four vertices $\{p, q, x_{i-1}^+, x_i^-\}$ in the required way if x is within the triangle $\{p, x_{i-1}^+, x_i^-\}$ and above the extension of the edge (q, x_i^-) into that triangle.



■ **Figure 7** Adding singletons near edge (x_{i-1}^+, x_i^-) if it is (left) vertical with $s_i = 7$, or (right) on the lower-left diagonal with $s_i = 4$. For clarity, not all singletons are shown.

Let P be the set of points that are one unit left of the drawing of (x_i^-, x_{i-1}^+) , ends included. We have $|P| = s_i + 6$ by our construction. Edge (p, x_{i-1}^+) has slope less than 2.9, so at most 3 points of P are above (p, x_{i-1}^+) . Edge (p, x_i^-) has positive slope, so all points of P are above (p, x_i^-) . Edge (q, x_i^-) has slope more than -2.9 (by a symmetric argument), so at most 3 points of P are below the extension of (q, x_i^-) . This leaves at least s_i points. We use the top points for the singletons of C_i (i.e., connecting to x_{i-1}^+) and the bottom points for the singletons of C_{i-1} (i.e., connecting to x_i^-). The total number of crossings created matches the number achieved in Section 3.

Case 2: $x_i^-, x_{i-1}^+ \in \mathcal{G}_L$ (the case $x_i^-, x_{i-1}^+ \in \mathcal{G}_R$ is symmetric): Edge (x_i^-, x_{i-1}^+) is drawn with slope 1. Let p, q be the two vertices in $T(C_i) \cap T(C_{i-1})$, say $p \in \mathcal{G}_T$, and $q \in \mathcal{G}_R$.

Let P be the set of grid points that are one unit left of the drawing of (x_i^-, x_{i-1}^+) excluding the lowest such grid point. We have $|P| = s_i + 4$ by construction. Edge (x_{i-1}^+, p) has slope more than $\frac{10}{7}$, while the line from x_{i-1}^+ to the fourth point from the left of P has slope $\frac{4}{3} < \frac{10}{7}$, so at most 3 points of P are left of edge (x_{i-1}^+, p) . Edge (x_i^-, p) has slope > 1 , so all points of P are left of (p, x_i^-) . Edge (q, x_i^-) has slope less than $\frac{1}{5}$, while the line from x_i^- to the second point from the right of P has slope $\frac{1}{2}$, so only 1 point of P is above the extension of (q, x_i^-) . This leaves at least s_i points in P that are inside the face and can see q while only crossing (x_i^-, x_{i-1}^+) . We use the bottom points for single-cluster vertices of C_i (i.e., connecting to x_{i-1}^+) and the top points for single-cluster vertices of C_{i-1} (i.e., connecting to x_i^-). The total number of crossings created again matches the number achieved in Section 3.

All singletons are placed in an inner face of the drawing. The size of the drawing is thus determined by the coordinates of the vertices placed on the rays in the first step of the algorithm. This proves Theorem 6.

C Details for Theorem 14

It remains to argue how 2-connected components can be merged while maintaining straight-line drawings. For this, we show that one vertex can be forced to appear at the outer-face.

► **Lemma 25.** *Let G be a graph with a path decomposition \mathcal{P} of width 3, and let p be a vertex in bag X_1 . Then there exists a straight-line drawing of G with at most $2cr(G)$ crossings that has p on the convex hull.*

Proof. Convert \mathcal{P} into an alternating path decomposition; this can be done while keeping p in the first bag. We prove the claim by induction on the number of 2-connected components; in the base case (no cut-vertex) the claim holds by Lemma 13. If G has a cut-vertex v , then let G_1, \dots, G_k be the cut-components of v , named such that G_1 contains p . Recursively obtain a drawing \mathcal{D}_1 of G_1 that has p on the convex hull, using the induced path decomposition.

Consider $i \geq 2$ and the path decomposition \mathcal{P}_i of G_i induced by \mathcal{P} . If v happens to be in the first bag of \mathcal{P}_i , then draw G_i recursively with v on the convex hull, and merge (after an affine transformation) the result in the vicinity of the drawing of v in \mathcal{D}_1 .

If \mathcal{P}_i does not contain v in its first bag, then we modify it. Let X_j be the first bag of \mathcal{P} that does contain v , and let X_h be any bag with $h < j$ that contains vertices of G_i . Within G_1 there exists a path P from p to v , hence from X_1 to X_j , hence X_h contains at least one vertex of P . Since $v \notin X_h$, X_h must contain at least one vertex of $G_1 - \{v\}$, i.e., not in G_i . Hence in \mathcal{P}_i we have $|X_h| \leq 3$ and can add v to this bag. Doing this for all X_h , we obtain a path decomposition of G_i that has v in its first bag and that is still alternating. ◀

D Proof of Lemma 19

We want to show that in any good drawing of a maximum pathwidth- \mathbf{w} -graph any crossing belongs to at most $\mu := 2\mathbf{w} - 5$ cluster bicliques. Let $\chi := \{x_1, x_2, x_3, x_4\}$ in age order be the four distinct endpoints of edges involved in a specific crossing. For any cluster C whose biclique $K(C)$ may contain this crossing, we have $\chi \subseteq V(C)$ and $|T(C) \cap \chi| = 2$, since $K(C)$ is bipartite. Let X_i be the bag where x_4 (the youngest of χ) is introduced. Let X_k be the first size- \mathbf{w} bag where one of χ (say x') has been forgotten. We have two cases:

Case 1: $k < i$, i.e., vertex x' is forgotten before x_4 is introduced. All bags containing x' are X_{i-2} or before, and all bags containing x_4 are X_i or after. Any cluster C that uses χ must hence contain X_{i-1} , a \mathbf{w} -sized bag. Observe that any bag X belongs to at most $|X| - 2$ clusters since, starting with the oldest three vertices of X as anchor-triplet, each next cluster containing X forgets one of the anchor vertices and adds one other vertex of X to obtain its anchor-triplet. Hence there are at most $|X_i| - 2 = \mathbf{w} - 2 \leq 2\mathbf{w} - 5$ clusters containing χ .

Case 2: $i \leq k$. All bags between X_i and X_{k-1} contain all vertices χ . Consider a cluster C that uses the crossing, and let X_h be the oldest bag of C . Since x_4 must belong to C , we have $h \geq i$. We have two subcases:

- Assume first that $h \geq k$. Then the size- \mathbf{w} bag X_k belongs to C . As argued above bag X_k belongs to at most $|X_k| - 2$ clusters, so there are at most $|X_k| - 2 = \mathbf{w} - 2$ cluster using the crossing with $h \geq k$.
- Now assume that $h < k$, which by $h \geq i$ means that X_h contains all of χ . Recall that the anchor-triangle $T(C)$ is defined to be the three oldest vertices in X_h . Since $\chi \subseteq X_h$ and $|T(C) \cap \chi| = 2$, it follows that neither x_3 nor x_4 can be in $T(C)$. Therefore at least one anchor-vertex of C is older than x_4 , which means that C starts to the left of X_i . Also, the anchor-triangle of cluster C uses one of the $\mathbf{w} - 1$ vertices in $X_i - \{x_3, x_4\}$. We hence have at most $\mathbf{w} - 3$ clusters C that fall into this case.

Putting the two bounds together, we have at most $2\mathbf{w} - 5$ cliques that use a crossing.

We can show that this bound is tight. Figure 8 shows an example of a path decomposition of width $\mathbf{w} = 5$ for which the vertex set $\chi = \{4, 5, 8, 9\}$ belongs to $5 = 2\mathbf{w} - 5$ clusters, all of

which have exactly two vertices of χ in their anchor triangle.

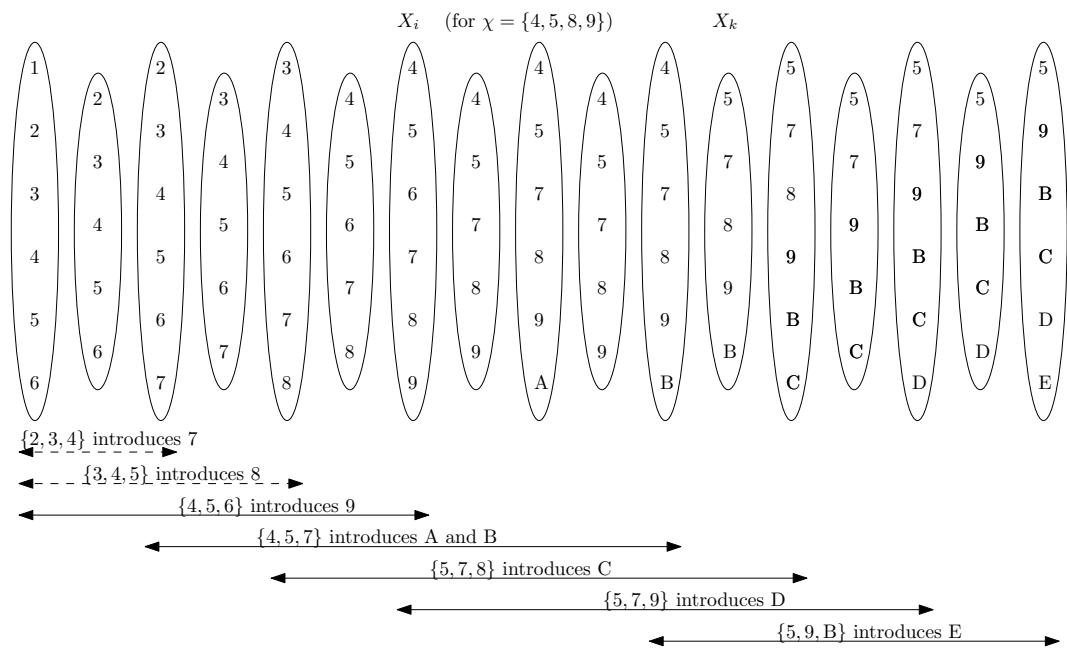


Figure 8 A path decomposition of width 5 with clusters.